

Reg. No.

--	--	--	--	--	--	--	--	--



MTH 553

IV Semester M.Sc. Degree Examination, September/October 2022
(CBCS – New Syllabus)
MATHEMATICS
Functional Analysis

Time : 3 Hours

Max. Marks : 70

- Note :** 1) Answer **any five full** questions.
2) Answer to **each full** question shall **not** exceed **eight** pages of the answer book. **No** additional sheets will be provided for answering.
3) Use of scientific calculator is **permitted**.

1. a) Let N be a normed linear space and let M be a closed linear subspace of N . For $x + M \in N/M$, define $\|x + M\| = \inf \{\|x + m\| : m \in M\}$. Show that N/M is a normed linear space. Further show that if N is a Banach space, then so is N/M .
- b) Prove that the interior of a proper linear subspace of a normed linear space is empty. **(10+4)**
2. a) Let $T : N \rightarrow N'$ be a linear transformation of normed linear spaces. Show that the following are equivalent :
- i) T is continuous
 - ii) T is continuous at the origin
 - iii) there exists a $K \geq 0$ such that $\|T(x)\| \leq K\|x\|$ for all $x \in N$.
 - iv) if $S = \{x \in N : \|x\| \leq 1\}$, then $T(S)$ is a bounded subset of N' .
- b) Give an example of a linear transformation of normed linear spaces which is not continuous.
- c) Let M be a closed linear subspace of a normed linear space N and let $x_0 \in N - M$. Prove that there exists a functional $f_0 \in N^*$ such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. **(6+4+4)**

P.T.O.



3. a) Let M be a linear subspace of a normed linear space N and let f be a functional defined on M . If x_0 is a vector not in M and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0 , then show that f can be extended to a functional f_0 on M_0 such that $\|f_0\| = \|f\|$.
- b) Let a Banach space B be made into a Banach space B' by means of a new norm. Show that the topologies generated by these norms are same if either is stronger than the other. **(12+2)**
4. a) Let B and B' be Banach spaces and let $T : B \rightarrow B'$ be a linear transformation. Then show that T is continuous if its graph is a closed set in $B \times B'$.
- b) State and prove the Uniform Boundedness theorem. Hence prove that, a non-empty subset X of a normed linear space N is bounded if and only if $f(X)$ is a bounded set of scalars for every $f \in N^*$. **(5+9)**
5. a) State and prove the Schwarz inequality for a Hilbert space H and hence derive that the inner product on H is jointly continuous.
- b) Is the Banach space $C([0, 1], \mathbb{R})$ of all continuous real functions on $[0, 1]$ with sup norm, a Hilbert space ? Justify.
- c) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M + N$ is also closed.
- d) If S is a non-empty subset of a Hilbert space H , then show that $S^\perp = S^{\perp\perp\perp}$. **(5+3+4+2)**
6. a) If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if $x \in H$, then prove that $x - \sum \langle x, e_i \rangle e_i$ is orthogonal to e_j for all j .
- b) Show that every orthonormal set in a Hilbert space H is contained in some complete orthonormal set. Use it to prove the following :
If M is a proper closed linear subspace of H , then there exists a non-zero vector z_0 in H such that $z_0 \perp M$. **(10+4)**
7. a) Prove that the following statements are equivalent for an orthonormal set $\{e_i\}$ of a Hilbert space H :
- $\{e_i\}$ is complete.
 - $x \perp e_i$ for all i implies $x = 0$.
 - If x is an arbitrary vector in H , then $x = \sum \langle x, e_i \rangle e_i$.
 - If x is an arbitrary vector in H , then $\|x\|^2 = \sum |\langle x, e_i \rangle|^2$.
- b) Prove that every Hilbert space is reflexive.
- c) Show that an operator T on a Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Hence prove that if N is a normal operator on H , then $\|N^2\| = \|N\|^2$. **(6+4+4)**



8. a) Let M be a closed linear subspace of a Hilbert space H , P be a projection on M and T be an operator on H . Then prove the following :
- i) M is invariant under T if and only if $TP = PTP$.
 - ii) M reduces T if and only if $TP = PT$.
- b) If P and Q are projections on closed linear subspaces M and N of a Hilbert space H , then prove that the following statements are equivalent :
- i) $P \leq Q$
 - ii) $\|Px\| \leq \|Qx\|$ for every $x \in H$
 - iii) $M \subseteq N$
 - iv) $PQ = P$
 - v) $QP = P$.
- c) Let $P_1, \dots, P_n$ be projections on the closed linear subspaces $M_1, \dots, M_n$ of a Hilbert space H and $M = M_1 + \dots + M_n$. Then show that $P = P_1 + \dots + P_n$ is a projection if and only if P_i 's are pairwise orthogonal and in this case P is the projection on M . (4+4+6)
-