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MTS 455

II Semester M.Sc. Degree Examination, September/October 2022
(CBCS – New Syllabus)
MATHEMATICS
Linear Algebra – II

Time : 3 Hours

Max. Marks : 70

Note : 1) Answer **any five full** questions.

2) Answer to **each full** question shall **not** exceed **eight** pages of the answer book. **No** additional sheets will be provided for answering.

3) **Use** of scientific calculator is **permitted**.

1. a) Define a bilinear form. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in the vector space $\mathbb{R}^2$, define $B(x, y) = x_1y_2 - x_2y_1$. Is B a bilinear form ? Justify.
- b) Define the matrix of a bilinear form. If A and A' are two matrices of a bilinear form with respect to two bases B and B' , then derive the relation between A and A' in terms of the matrix of change of basis.
- c) Prove that the bilinear form represented by the matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is positive definite if and only if $a > 0$ and $ad - b^2 > 0$. **(4+6+4)**
2. a) Prove that the following statements are equivalent for an $n \times n$ real matrix :
- i) A is symmetric and positive definite.
- ii) The form X^tAY represents dot product, w.r.t. some basis of $\mathbb{R}^n$.
- iii) $A = P^t P$ for some $P \in GL_n(\mathbb{R})$.
- b) Prove that the eigen values of a Hermitian matrix are real numbers.
- c) If a Hermitian form on V is not identically zero, then show that there is a vector $v \in V$ such that $\langle v, v \rangle \neq 0$. **(9+2+3)**

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3. a) Prove the existence of an orthonormal basis in a finite dimensional vector space V with a positive definite Hermitian form.
- b) Let $\langle, \rangle$ be a symmetric form on a real vector space V and let W be a subspace of V on which $\langle, \rangle$ is non-degenerate. If $(w_1, w_2, \dots, w_n)$ is a basis for W , then obtain the projection formula for V onto W .
- c) Let $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$. Determine the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto W . (5+5+4)
4. a) Define a Unitary operator on a Hermitian space V . Show that $T : V \rightarrow V$ is unitary if and only if $\langle Tv, Tw \rangle = \langle v, w \rangle \forall v, w \in V$.
- b) State and prove Spectral theorem for normal operators. (4+10)
5. a) Let $f : V \rightarrow V'$ be a surjective homomorphism of R -modules, with kernel W . Show that there is a one to one correspondence between submodules of V' and submodules of V containing W .
- b) Define a finitely generated module and a free module. Are all finitely generated modules free ? Justify. (9+5)
6. a) Let R be a non-zero ring. Prove that any two bases of the same free module over R have the same cardinality.
- b) Show that any subgroup of a free abelian group is free. (7+7)
7. a) Let V be an R -module. Prove that every submodule of V is finitely generated if and only if submodules of V satisfies ascending condition in the inclusion order.
- b) Let $\phi : V \rightarrow V'$ be a homomorphism of R -modules. If the kernel and image of ϕ are finitely generated, then show that V is finitely generated. (7+7)
8. a) State and prove Hilbert basis theorem.
- b) Find matrix P such that $P^{-1}AP$ is a diagonal matrix, where
- $$A = \begin{bmatrix} t^2 - 3t + 1 & t - 2 \\ (t - 1)^3 & t^2 - 3t + 2 \end{bmatrix}. \quad (10+4)$$
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