

Reg. No.

--	--	--	--	--	--	--	--	--	--

**MTS 456**

II Semester M.Sc. Degree Examination, September/October 2022
MATHEMATICS
Ordinary Differential Equations
(CBCS – New Syllabus)

Time : 3 Hours

Max. Marks : 70

Note : 1) Answer **any five full** questions.2) **No** additional sheets will be provided for answering.3) **Use** of scientific calculator is **permitted**.

1. a) If ϕ is any solution of $L_n(y) = y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$ on an interval I containing a point x_0 , then prove that $\forall x$ in I , $e^{-K|x-x_0|} \|\phi(x_0)\| \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{K|x-x_0|}$ where, $\|\phi(x)\| = [\phi(x)^2 + |\phi'(x)|^2 + \dots + |\phi^{(n-1)}(x)|^2]^{1/2}$ and $K = 1 + |a_1| + |a_2| + \dots + |a_n|$.
 b) Let $\phi_1(x)$ and $\phi_2(x)$ be linearly independent solutions of $L_2(y) = 0$ on an interval I . Then prove that every solution of $L_2(y) = 0$ can be expressed uniquely as $\phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$, where c_1 and c_2 are constants.
 c) Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any n constants and let x_0 be any real number. Prove that on any interval I containing x_0 there exists at most one solution ϕ of $L_n(y) = 0$ satisfying $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \dots, \phi^{(n-1)}(x_0) = \alpha_n$. **(7+4+3)**
2. a) Compute the Wronskian of $y''' - 4y' = 0$. Verify that they are linearly independent.
 b) Use method of variation of parameters to find a particular solution of the equation
 i) $x^2 y'' - 2xy' + 2y = 6x^4$
 ii) $y'' + y = \sec x$.
 c) Find the solution ϕ of the initial-value problem
 $y''' + y = 0, y(0) = 0, y'(0) = 1, y''(0) = 0$. **(4+4+6)**
3. a) Solve the following equation by the method of undetermined coefficients :
 i) $y'' - 2y' + y = 2e^x + 2x$
 ii) $y'' + y = xe^x \cos 2x$.
 b) If ϕ_1 is a solution of $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, on an interval I and $\phi_1(x) \neq 0 \forall x \in I$, then show that $\phi_2(x) = \phi_1(x) \int_{x_0}^x \frac{1}{[\phi_1(s)]^2} \text{Exp} \left[\int_{x_0}^s \frac{-a_1(\xi)}{a_0(\xi)} d\xi \right] ds$ is another solution. **(8+6)**

P.T.O.



4. a) Find the power series solution of $y'' + 3xy' + 3y = 0$, $y(0) = 2$, $y'(0) = 3$.

b) Prove the following recurrence relations :

$$\text{i) } (2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$$

$$\text{ii) } nP_n = xP'_n - P'_{n-1}$$

where $P_n(x)$ is the Legendre polynomial.

(8+6)

5. a) Show that $\int_{-1}^1 P_n(x)P_m(x)dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2m+1}, & m = n \end{cases}$ where $P_n(x)$ represents the Legendre polynomials.

b) Obtain the general solution of $(x^2 + 1)y'' + xy' - xy = 0$ about regular singular point $x = 0$.

(7+7)

6. a) Using Frobenius method, find the series solution of $2x^2y'' + x(2x + 1)y' - y = 0$ near the singular point $x = 0$.

b) Prove that $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$.

(10+4)

7. a) State and prove existence and uniqueness theorem for the solution of the initial value problem $y' = A(x)y$, $y(x_0) = y_0$.

b) Solve the system of equations $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

(7+7)

8. a) Let f be a continuous real valued function on the rectangle $R : |x - x_0| \leq a$, $|y - y_0| \leq b$, ($a, b > 0$) and let $|f(x, y)| \leq M$ for all (x, y) in R . Further suppose that f satisfies a Lipschitz condition with constant K in R . Then prove that the successive approximations

$$\phi_0(x) = y_0, \phi_{k+1}(x) = y_0 + \int_{x_0}^x f(t, \phi_k(t)) dt, k = 0, 1, 2, \dots$$

converges on the interval $I : |x - x_0| \leq \alpha = \min\left\{a, \frac{b}{M}\right\}$ to a solution ϕ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$ on I .

b) Apply Picards method to solve the IVP upto 3rd approximation $\frac{dy}{dx} = x + y^2$ given that $y = 0$ at $x = 0$.

(9+5)